ANALYTICAL SELECTION OF PARAMETERS IN LINEAR CONTROL SYSTEM DESIGN

by

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UNITED STATES NAVAL POSTGRADUATE SCHOOL

THESIS

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ABSTRACT

Fixed configuration feedback control often forms the basis for the design of a linear control system. The design freedom is limited to the adjustment of the free parameters of the system. Analytical methods prove to be valuable in these circumstances and the aim is the optimization of a selected index of performance. A suggested index is developed for a regulator system from a comparison of the dynamics of the subject system with those of a desired model along a transient state space trajectory. The result is an optimal system dependent on the initial state chosen for the transient response. The effects of the initial state on the optimal system are investigated and the final selection is based on statistical considerations.

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INTRODUCTION

The design of feedback control systems sometimes involves the selection of free parameters in the plant and/or compensator in order to satisfy the requirements of the performance specifications. The configuration of the system is assumed to be fixed, i.e., a compensator has already been chosen and the design freedom is limited to the adjustment of one or more parameters. The trial and error process for the selection of several parameters represents a very laborious and sometimes impossible task for the classical design by root locus methods. Analytical methods are appropriate in this case. A measure of quality is established for the system in question by the construction of an index of performance, which upon minimization (or maximization) will yield the optimum values for the free parameters of the system.

Different criteria could be adopted for the construction of the index of performance, all normally based on physical reasons and each satisfying particular requirements. Many performance indices originating from good engineering reasoning lead to analytically insolvable problems. A compromise in the selection of the index is then necessary to obtain a solution. The classical forms of the performance specifications, such as rise time, maximum overshoot, settling time, etc., in the time domain or bandwidth, phase and gain margins, etc., in the frequency domain, are transformed into a single measure by conceiving an ideal model

whose behavior meets all the performance specifications. A mathematical relationship established between some variables of this model and the corresponding variables in the actual system serves as a measure of quality as the free parameters are changed in a search for the optimum adjustment. The time integral of the difference-squared between the output of the model and the output of an actual system is a practical index of performance when the system is subjected to transient input signals. Here the model has a transfer function equal to unity. This is usually solved with the aid of Parseval's Theorem[2].

An index of performance has been derived for the regulator problem[1]. This index of performance, or cost function, is a quadratic function of the state vector,

$$J = \int_0^\infty \underline{x}^T Q \, \underline{x} \, dt \tag{1}$$

The matrix of the quadratic form, Q, is the weighting matrix and contains the free and fixed parameters of the actual system and some weighting factors determined by the model or ideal system. The model obtained from the performance specifications is of the same order as the actual system. The weighting factors are adjusted in such a way that if all the parameters of the actual system were free, the cost function would exhibit an absolute minimum at the model values. If the actual system is constrained to have less than n independent free parameters, a minimization of the cost function will yield the optimum values for these

parameters.

The measure of quality is obtained by disturbing the system with some initial conditions. For a stable system, the state space trajectory ends up at the origin for large values of time. This type of excitation is characteristic of the regulator problem. By minimizing this cost function the trajectory of the subject system is forced to approximate as closely as possible the trajectory of the model. If all n parameters of the system were free, it would indeed simulate identically the dynamics of the model and hence satisfy in full agreement the performance specifications. In any other case, the model behavior will not be reproduced, but the actual system will attain its best performance as measured by this optimization criterion.

The importance of this method over the common types of indices is the consideration given to every part of the dynamics of the system and not to a single manifestation as, for example, the error in position. The contributions from all the state variables are properly weighted according to the characteristics of the model, which carries the information of the performance specifications.

The evaluation of the integral in equation (1) is done by Liapunov functions with vector-matrix methods, and the final form is shown to be a quadratic function of the initial state,

$$J = \underline{x}^T(0)P\underline{x}(0) \qquad (2)$$

The matrix of this quadratic form, P, is related to the

weighting matrix and the actual system's matrix.

The nature of this solution implies that the cost function will be different for different initial states and hence the optimum system will depend on the initial conditions chosen for the calculation.

In this paper, a similar index of performance is developed, based upon a different approach than that described above. To obtain a measure of performance, at every point on the state space trajectory of the actual system, a comparison is made between the dynamics of this system and the dynamics the model would have if it were at that particular point.

The index of performance can be expressed in the same form as equation (1) and hence is also a function of the initial state. However, the weighting matrix is now directly obtained from the coefficients of the model and actual system characteristic equations. This is a much more direct method than that for obtaining the Q matrix in equation (1), which relies on complicated algorithms for the weighting factors. In the analysis of this index it is found that the results are exactly the same as for the index of equation (1). Further investigation shows that both cost functions are intimately related by a constant factor and an additive constant. The effects on the optimum system of the initial state chosen for the calculation are investigated in this paper and the final decision on the design is based on the statistical characteristics of the disturbing initial conditions.

The new concept developed also suggests a time-varying optimization scheme by which the actual system would be made to follow exactly the state space trajectory of the model by continuously varying its free parameters so as to match at every point the dynamics of both systems. Examples are presented to complement the theory.

CHAPTER II

THE INDEX OF PERFORMANCE

The first step in the process of analytical design is to establish the concept on which the index of performance will be based. The objective of this performance index is to include in a single number a measure of quality for the performance of the system. Factors such as mathematical difficulties will perhaps force the designer to modify his performance index in favor of mathematical simplicity, hopefully keeping the essence of the idea that originally motivated it. Once the configuration of the system is fixed and the index of performance has been defined, the design proceeds in a straightforward manner. For the type of index to be considered here, the performance specifications are transformed into an ideal or model system thus fixing the location on the s-plane of the required zeros and poles of the transfer function. This model is of the same order as the actual system and fully satisfies the specifications. The model and actual systems can be represented by the signal flow graphs shown in Figure 1.

The feedback coefficients of the model are defined by the performance specifications. Some of the coefficients of the actual system are constrained by the physical nature of the plant, and the others are functions of the free parameters. If all coefficients in the actual system were free it would be trivial to reproduce the model. Generally this is not the case and the free parameters must be adjusted

according to some criterion.

A measure of quality which originates from a good physical concept is given by

$$J = \int_0^{\infty} \left[\frac{d^n y}{dt^n} - \frac{d^n x}{dt^n} \right]^2 dt \qquad (3)$$

where the integrand is the square of the difference between the n^{th} derivatives of the model and actual systems differential equations respectively, when the system is excited by initial conditions only. This index gives a comparison of the complete dynamics of both systems in terms of the highest derivatives, and does not require any weighting factors since it implies a direct relationship between the two systems. The minimization of equation (3) as function of the free parameters results in an actual system whose dynamic behavior is a close approximation to that of the model. However, the development of this performance index into a practical form presents the impediment of requiring the solution of both system differential equations, which shadows the merit of its physical meaning and renders the integration of equation (3) very difficult. It is then necessary to conceive a similar structure for the index of performance but one which permits convenient mathematical manipulations.

A similar way of evaluating the performance of the system can be obtained from a comparison along a trajectory, of the n^{th} derivative of the actual system differential equation with that of the model when both systems are in the

same state. It is again assumed that the system is driven by initial conditions only. The motivation for this index arises from a consideration of a second order process. Consider the system shown in Figure 2. The system differential equation is

$$\ddot{x} + a_2\dot{x} + a_1 x = 0 \tag{4}$$

where a_1 is the free parameter

The performance specifications require a model system characterized by the differential equation

$$\ddot{y} + g_2\dot{y} + g_1 y = 0 \tag{5}$$

The actual system trajectory in the phase plane is shown in Figure 3. At any point on this path, $\ddot{x}$ is determined by the coordinates of the phase plane and the coefficients of the differential equation

$$\ddot{x}_1 = -a_2\dot{x}_1 - a_1 x_1 \tag{6}$$

Also, equation (5) determines the value of $\ddot{y}$ at any point on the phase plane

$$\ddot{y}_1 = -g_2\dot{y}_1 - g_1 y_1 \tag{7}$$

where the subscript 1 indicates the particular value of the variables at any given point.

Along any trajectory of the actual system there will be a continuous function of the state giving the value of $\ddot{y}$. This function is

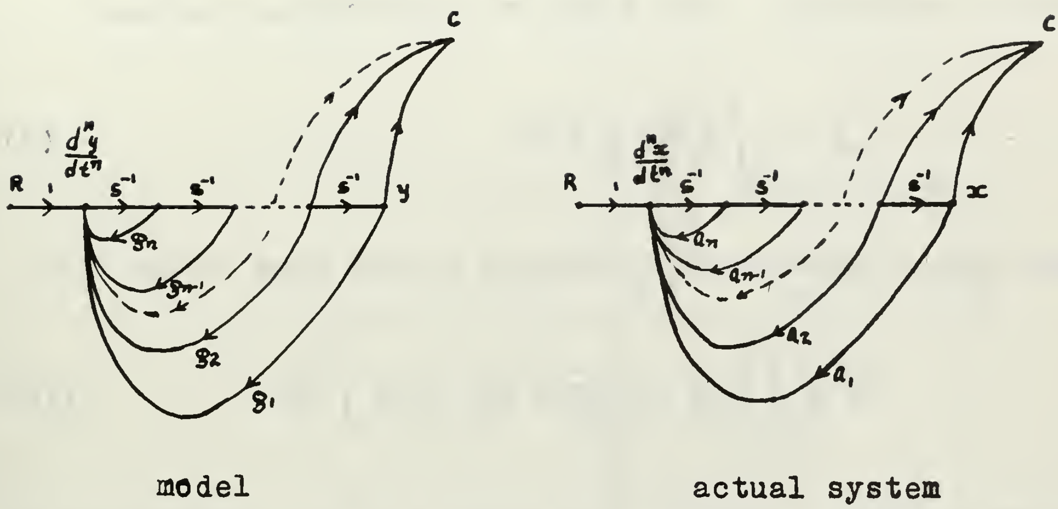

Fig. 1. Signal flow graphs for model and actual system

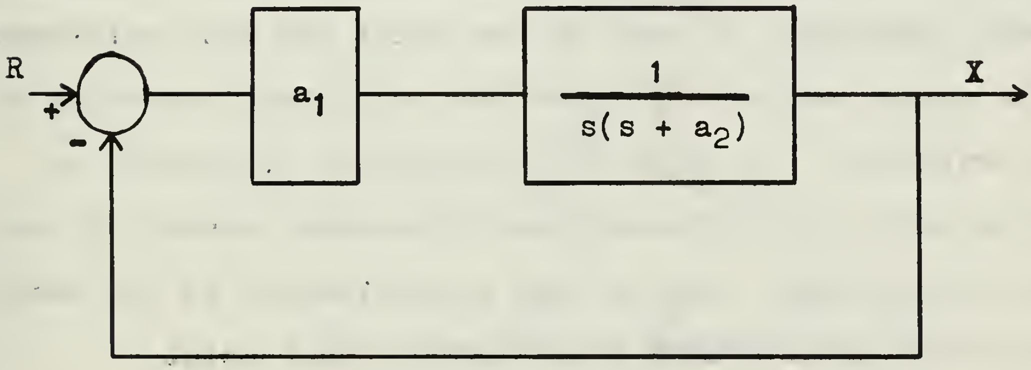

Fig. 2. A typical second order system

$$\ddot{y} = - g_2\dot{x} - g_1 x \qquad (8)$$

The optimization criterion is the minimization of the difference squared between the variables $\ddot{x}$ and $\ddot{y}$ along the actual system trajectory. The index of performance becomes

$$J = \int_0^\infty [\ddot{x} - \ddot{y}]^2 dt \qquad (9)$$

when actual system and model are in the same state. Then

$$J = \int_0^\infty \left[(g_2 - a_2)\dot{x} + (g_1 - a_1)x\right]^2 dt \qquad (10)$$

Figure 4 shows a signal flow graph for the calculation of the index.

In searching for the optimum adjustment of a_1, the trajectory along which this comparison is made will change until the value of the index is a minimum. This will be the closest trajectory to that of the model and the performance of the actual system will have been optimized according to this criterion. In terms of the dynamics of motion, we wish to make the acceleration of the actual system at any point on the phase plane a best approximation to the acceleration the ideal system would have at that point.

This idea is extended to an n^{th} order system

$$\frac{d^n x}{dt^n} + a_n \frac{d^{n-1}x}{dt^{n-1}} + \cdot \cdot \cdot + a_2 \frac{dx}{dt} + a_1 x = 0 \qquad (11)$$

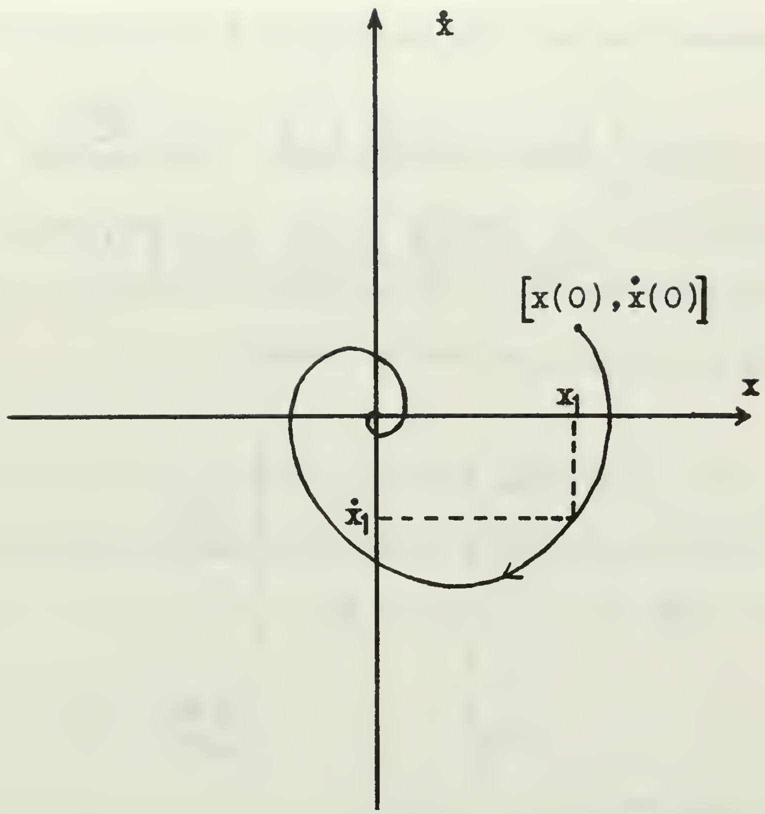

Fig. 3. Phase plane trajectory for system of Fig. 2

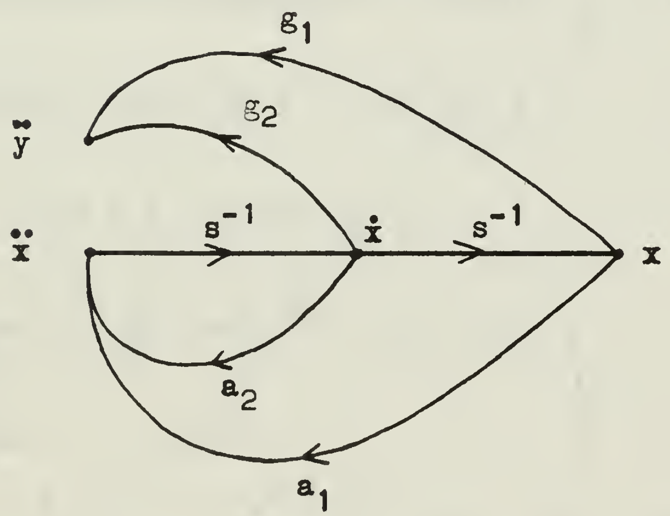

Fig. 4. Signal flow graph showing the structure of the index

where the index of performance becomes

$$J = \int_0^\infty \left[\frac{d^n x}{dt^n} - \frac{d^n y}{dt^n} \right]^2 dt \tag{12}$$

when actual system and model are in the same state. Then

$$J = \int_0^\infty \Big[(g_n - a_n)\frac{d^{n-1}x}{dt^{n-1}} + (g_{n-1} - a_{n-1})\frac{d^{n-2}x}{dt^{n-2}} + \cdots + (g_2 - a_2)\frac{dx}{dt} + (g_1 - a_1)\,x \Big]^2 dt \tag{13}$$

By defining the state vector

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x \\ \frac{dx}{dt} \\ \vdots \\ \frac{d^{n-1}x}{dt^{n-1}} \end{bmatrix} \tag{14}$$

this index can be written in quadratic form

$$J = \int_0^\infty \Big[(g_n - a_n)^2 x_n^2 + (g_{n-1} - a_{n-1})^2 x_{n-1}^2 + \cdots + (g_1 - a_1)^2 x_1^2 + 2(g_n - a_n)(g_{n-1} - a_{n-1})\,x_n x_{n-1} + \cdots + 2(g_2 - a_2)(g_1 - a_1)\,x_2 x_1 \Big]\, dt \tag{15}$$

$$= \int_0^\infty [x_1 \;\; x_2 \;\cdots\; x_n] \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & \ddots & \vdots \\ b_{n1} & \cdots & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{16}$$

where $b_{ij} = (g_i - a_i)\,(g_j - a_j)$, $i = 1,2,\ldots..n$ and $j = 1,2,$ ....n.

Then

$$J = \int_0^{\infty} \underline{x}^T B \, \underline{x} \, dt \tag{17}$$

where the matrix of the quadratic form, B, is a positive semi-definite, symmetric matrix.

This is the general form of the index of performance as given in Equation (1). It is solved by vector-matrix methods using Liapunov's Theorem[1]. The solution is

$$J = \underline{x}^T(0) \; P \; \underline{x}(0) \tag{18}$$

where P is the matrix of the Liapunov function and is related to the weighting matrix B and the actual system matrix, A_a, by

$$-B = A_a{}^T P + P A_a \tag{19}$$

The B matrix is directly obtained from the model and actual system coefficients as shown in equation (16). The actual system matrix is defined by the matrix differential equation in canonical phase variable form

$$\dot{\underline{x}} = A_a \, \underline{x} \tag{20}$$

where

$$A_a = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \\ \vdots & & & & 1 \\ -a_1 & -a_2 & \cdots & & -a_n \end{bmatrix} \tag{21}$$

Signal flow graphs are helpful in solving equation (19) for the elements of the P matrix[1]. The results for a second order system are stated here for illustration.

$$p_{11} = -b_{12} + \frac{a_2}{2a_1} b_{11} + \frac{1}{2a_2} b_{11} + \frac{a_1}{2a_2} b_{22}$$

$$p_{12} = \frac{1}{2a_1} b_{11} = p_{21} \qquad (22)$$

$$p_{22} = \frac{1}{2a_2} b_{22} + \frac{1}{2a_1a_2} b_{11}$$

The cost function, equation (18) becomes

$$J = x^2_1(0)p_{11} + 2x_1(0)x_2(0)p_{12} + x_2^2(0)p_{22} \qquad (23)$$

The minimization of the cost function will yield the optimum adjustment for the free parameters. Because of the nature of this artifice, the optimum system will depend on the initial state chosen for the calculation. This problem is discussed in Chapter III.

A TIME-VARYING OPTIMIZATION METHOD

The minimization of the index of performance developed in the last section produces a set of parameters for which the dynamics of the system is a best approximation to that of the model. In terms of the state space, this means that the actual system trajectory is now the closest trajectory to that of the model which can be achieved with this constrained system under the effect of a given set of initial conditions. The index is now a measure of the total error incurred by the system in going from this particular initial state to the origin of the state space. Because it is a

quadratic function, it gives more importance to large errors than to small errors but is not capable of reducing the magnitude of any particular error in performance.

The criterion on which this index is based suggests a possible way to obtain with the actual system an exact reproduction of the model dynamics if the free parameters are allowed to vary continuously according to some function of the present state. Actually, it would be enough to have only one free parameter in an n^{th} order system to implement this scheme.

Consider again the second order system of Figure 2. The actual system and model differential equations are given in equations (4) and (5) respectively. When both systems are in the same state.

$$y = x$$
$$\dot{y} = \dot{x}$$

and the condition for the two state space trajectories to be the same (and so the dynamics of both systems) is

$$\ddot{y} = \ddot{x}$$

or

$$g_1 x + g_2 \dot{x} = a_1 x + a_2 \dot{x}$$

then

$$a_1 = g_1 + (g_2 - a_2) \frac{\dot{x}}{x} \qquad (24)$$

Equation (24) is the required functional relation between the free parameter and the state of the system. This is plotted in Figure 5 for a typical case.

The curve shows a discontinuity whenever the trajectory crosses the $\dot{x}$-axis in the phase plane. Since infinite gain is not practically feasable, the function can be approximated as shown in Figure 5. A change in the sign of the gain factor from time to time is also required. The reason for the discontinuities is that when $x = 0$, the free parameter, a_1, which modifies the variable x in the feedback path, has no effect on the behavior of the system, now solely controlled by the variable $\dot{x}$ and its fixed feedback parameter a_2. Thus, an infinite gain is the result of the impossibility of altering the system's trajectory.

For the general case of an n^{th} order system, the condition under which the actual system will duplicate the model dynamics is

$$g_1 x + g_2 \frac{dx}{dt} + \cdots + g_n \frac{d^{n-1}x}{dt^{n-1}} =$$

$$a_1 x + a_2 \frac{dx}{dt} + \cdots + a_n \frac{d^{n-1}x}{dt^{n-1}} \tag{25}$$

If we assume that a_1 is the only one free parameter, then

$$a_1 = g_1 + (g_2 - a_2)\frac{dx/dt}{x} + \cdots + (g_n - a_n)\frac{d^{n-1}x/dt^{n-1}}{x} \tag{26}$$

Equation (26) shows that only one free parameter is needed to reproduce the model dynamics if the state of the system is known. Normally the fact that the state vector is completely measurable implies that the feedback paths

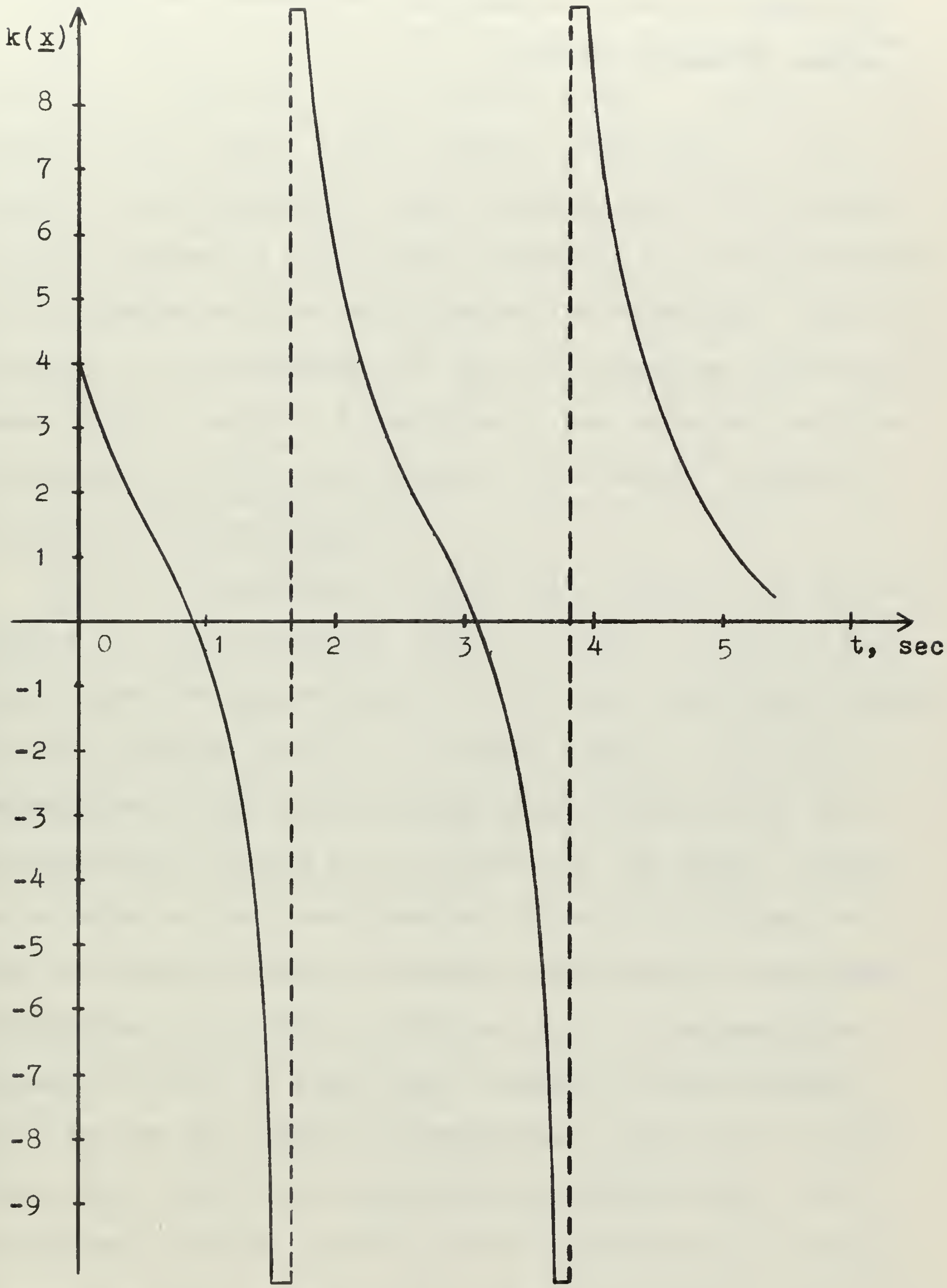

Fig. 5. A plot of $k(\underline{x})$ versus t; model: $\ddot{y} + 2.8\dot{y} + 4y = 0$
actual system: $\ddot{x} + \dot{x} + kx = 0$ and $\underline{x}^T(0) = (1 \quad 0)$

can be modified to copy the model characteristics exactly. This discussion would apply in those special cases in which the state vector is measurable but cannot be used directly in the feedback channel.

CHAPTER III

THE OPTIMUM SYSTEM AS A FUNCTION OF THE INITIAL STATE

The index of performance that will yield the optimum system is a function of the initial state. We wish to investigate the effect of the initial conditions on the optimum pole locations. These optimum poles will occupy certain regions on the s-plane according to the constraints of the system and the requirements for stability. Also of interest is the knowledge of how the extension and shape of these regions vary as a function of the relative position of the model poles with respect to the actual system's attainable root regions.

Only the normalized initial state vector will be considered in this discussion since all other points in the state space are proportional to this unit vector and differ in cost function only by a constant factor. For each orientation of the unit initial state vector there is a corresponding optimum set of poles for the actual system and a value of the cost function which is a minimum for this particular choice of initial conditions. Any other orientation will yield a different set of optimum poles. However, it will be seen that, because of the symmetry exhibited by the index of performance, only a half of the state space need to be searched for optimization. The design must provide a system whose performance is optimum in some modified sense for all possible initial conditions.

The selection of such a set of optimum parameters must

be done on the basis of the statistical properties of the disturbance. If the probability density function can be established, then the average value of the cost incurred by the system can be found for all sets of optimum parameters. The set that provides the minimum average cost is the design selection. This, of course, implies the consideration of the behavior of the control system as a regulator. It is not possible to obtain an analytical relationship for the optimum poles as a function of the initial state and the proximity of the desired model poles for a given constrained actual system. However, the general trend is that the regions containing the optimum poles reduce, and finally collapse to a point, when the model poles get closer to and reach the permissible regions for the actual system. Similarly, the same effect is obtained by gradually releasing the constraints of the actual system but keeping the same characteristics of the desired model. These effects will be discussed for the second and third order systems by means of examples.

SECOND ORDER SYSTEM

The results of the optimization scheme developed in Chapter II will now be discussed for a typical second order system. Two parameters define the dynamics of this system. The method is applicable when one of this parameters, usually in the plant, is fixed. With only one degree of freedom, the possible poles for this system must lie on a definite locus. A pair of required poles outside this locus

is simply unattainable and the optimization process must be employed.

Consider the second order system shown in Figure 6. The transfer function is

$$\frac{C}{R} = \frac{k}{s^2 + s + k} \tag{27}$$

The free parameter is the gain of the system whereas the plant contains the fixed parameter. The performance specifications require a model defined by the transfer function

$$\frac{C}{R} = \frac{4}{s^2 + 2.8s + 4} \tag{28}$$

It is noted that the model poles are unattainable by the actual system. It is then necessary to find an adjustment for k that will give the best performance with respect to the model according to the criterion previously established.

The cost function is given by equation (23)

$$J = x_1^2(0)p_{11} + 2x_1(0)x_2(0)p_{12} + x_2^2(0)p_{22} \tag{23}$$

The elements of the P matrix are those given in equation (22). The weighting matrix from equation (16) is

$$B = \begin{bmatrix} (k-4)^2 & (k-4)(1-2.8) \\ (1-2.8)(k-4) & (1-2.8)^2 \end{bmatrix} \tag{29}$$

then, J becomes

$$J = x_1^2(0)(0.5k^2-0.08k-3.2+\frac{8}{k}) + 2x_1(0)x_2(0)(0.5k-4+\frac{8}{k}) + x_2^2(0)(0.5k-2.38+\frac{8}{k}) \quad (30)$$

Only points on a unit circle in the initial condition plane need be investigated. Equation (23) is symmetric with respect to a diameter on the initial condition unit circle. As a further simplification, only the upper semicircle will be considered.

The root locus for this system is shown in Figure 7. For each value of θ, where θ is the orientation angle of the state vector around the upper unit semicircle of the initial conditions plane, Figure 7(a), a value of k_{opt} is found by minimizing the cost function with respect to this parameter. The plot of optimum regions for the corresponding poles of the actual system is shown in Figure 7(b) together with the desired model poles. The extremes are located at (-0.5+j1.936) and (-0.5+j0.775) for the upper half of the s-plane. Each pair of conjugate poles on this region will give an optimum performance only when the system is excited by the initial conditions that yielded that pair of poles. For any other initial state there will be a different set of optimum poles and a correspondingly different setting of the free parameter for optimum response. It is not possible, of course, to modify the adjustment of the free parameter according to the type of disturbance or even worse, to predict its values. It is then required to adopt a single

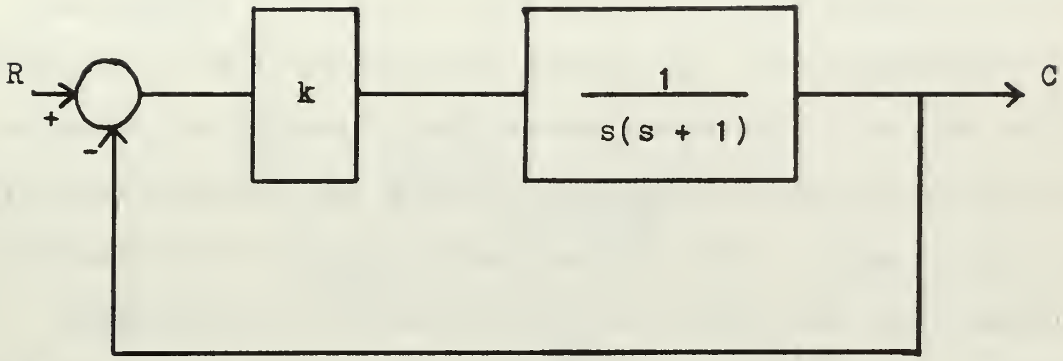

Fig. 6. A constrained second order system

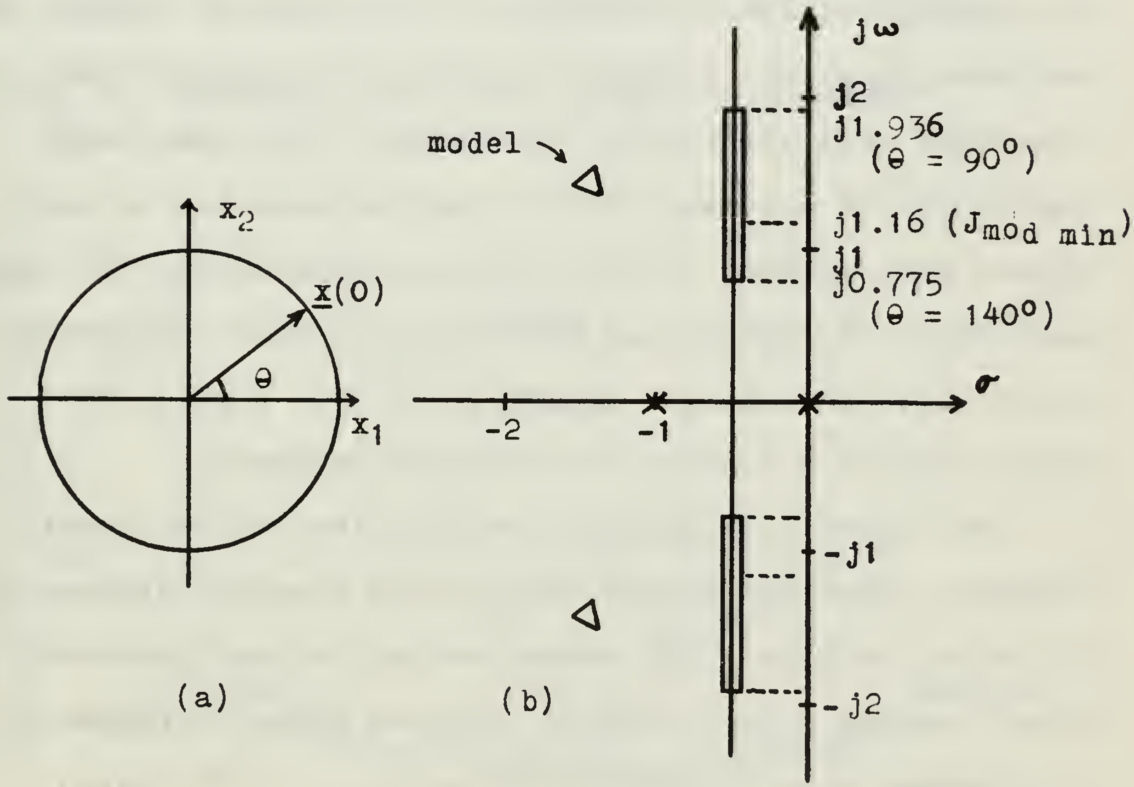

Fig. 7. (a) Initial condition plane, (b) Root locus for $s^2 + s + k = 0$

value for the free parameter that will give the best average index for all possible situations. Figure 8 shows the value of the minimum cost as a function of initial conditions. The horizontal axis represents the orientation of the initial state vector. This curve shows that a vector orientation of 140° will give an absolute minimum for the cost function of this example. This corresponds to $k_{opt} = 0.85$ and the optimum pole lies right at the bottom of the allowable region. This would seem the best adjustment to use in the actual system. Figure 9 shows the cost incurred by the system as a function of the initial state for different optimum settings of the free parameter. It is seen that the optimum parameter obtained at the absolute minimum of the cost function of Figure 8 yields the highest index when used with all initial conditions. The area under each curve is a measure of the cost accumulated by the system when excited by all initial conditions on the upper semicircle of Figure 7(a) for a given optimum adjustment of the free parameter. Figure 10 shows a plot of the total cost as a function of the free parameter.

The curve in Figure 8 also exhibits another local minimum, which for some models is the absolute minimum of the cost function. The corresponding optimum parameter also produces a high average index as shown in Figure 9. The minimum area is obtained for a $k_{opt} = 2.39$, which corresponds to an initial state vector orientation of 25°.

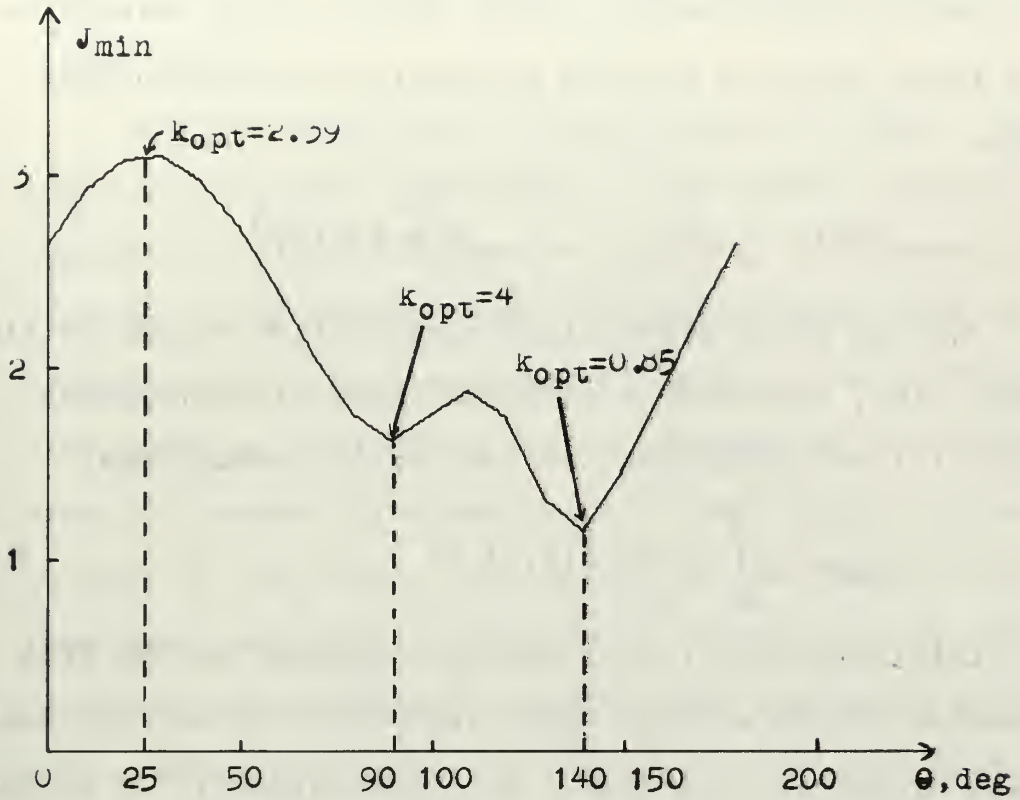

Fig. 8. Minimum cost versus initial conditions

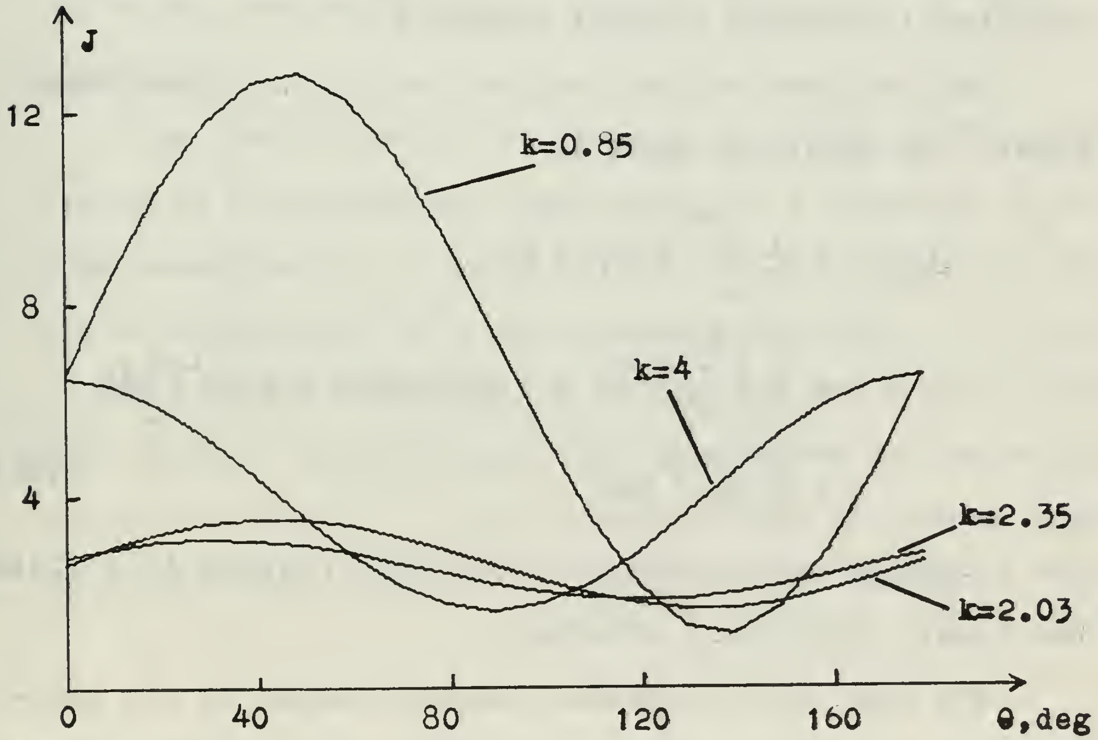

Fig. 9. Cost as a function of the initial state for different optimum settings of the free parameters

This particular value of k would be the design selection when there exists a uniform probability on the initial state

$$p(\theta) = \frac{1}{\pi}\, d\theta \;, \qquad\qquad 0^\circ \leq \theta \leq 180^\circ$$

For any given probability distribution on the initial state, $p(\theta)$, we define a modified index of performance, which will be independent of the initial conditions,

$$J_{mod} = \int_0^{\pi} p(\theta)\; J(\theta,k)\; d\theta \tag{31}$$

The minimization of this index with respect to the free parameter yields an adjustment for which the average cost of the system is a minimum. As a consequence, the maximum value of the cost for any possible initial state will also have been minimized. This is equivalent to a minimax solution in optimal control theory[3].

For the case of the uniform probability considered above, the modified index is

$$\begin{aligned} J_{mod} &= \frac{1}{\pi}\int_0^{\pi} J(\theta,k)\, d\theta \\ &= \frac{1}{\pi}\int_0^{\pi}\left[p_{11}\cos^2\theta + 2p_{12}\sin\theta\cos\theta + p_{22}\sin^2\theta\right]d\theta \\ &= \frac{1}{2}(p_{11} + p_{22}) \end{aligned} \tag{32}$$

The minimization of equation (32) with respect to k yields the result previously obtained.

The size of the optimum region depends on the proximity of the model poles to the root locus of the actual

system. Figure 11 shows how these regions increase as the model poles get further apart from the root locus of the actual system.

Only the upper half of the s-plane is shown. For model poles on the root locus of the actual system the regions are simply reduced to a point. The growth of the regions as the model poles get further apart from the actual system root locus is more accentuated in the positive $j\omega$ direction, an indication of the tendency of the index to make the natural frequency, ω_n , of the actual system equal to that of the model. The effect of the value of the imaginary part of the model poles on the size of the region is not pronounced, except that for small values of ω , some optimum poles could lie on the real axis. Only near model poles are of interest here since remote poles would probably require a change in the configuration of the actual system, which is outside the scope of this discussion.

The results stated so far are all based on the minimization of a performance index which is a function of the free parameter for a given set of initial conditions and can be represented by a two-dimensional curve. This curve is actually the intersection of a plane and a more general cost surface obtained when both parameters in the actual system are free[1]. If in equation (23) the actual system parameters are variable the cost function becomes

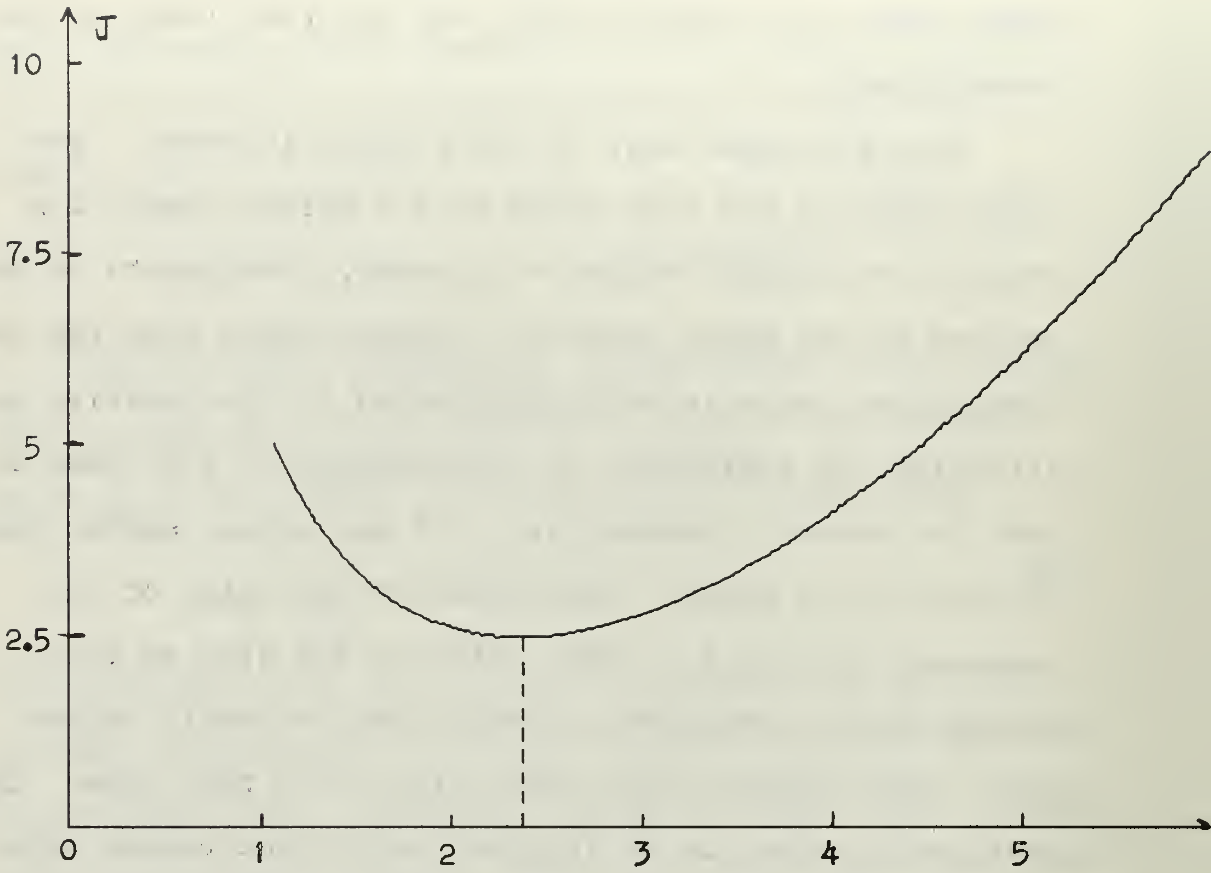

Fig. 10. Total cost as a function of the free parameter

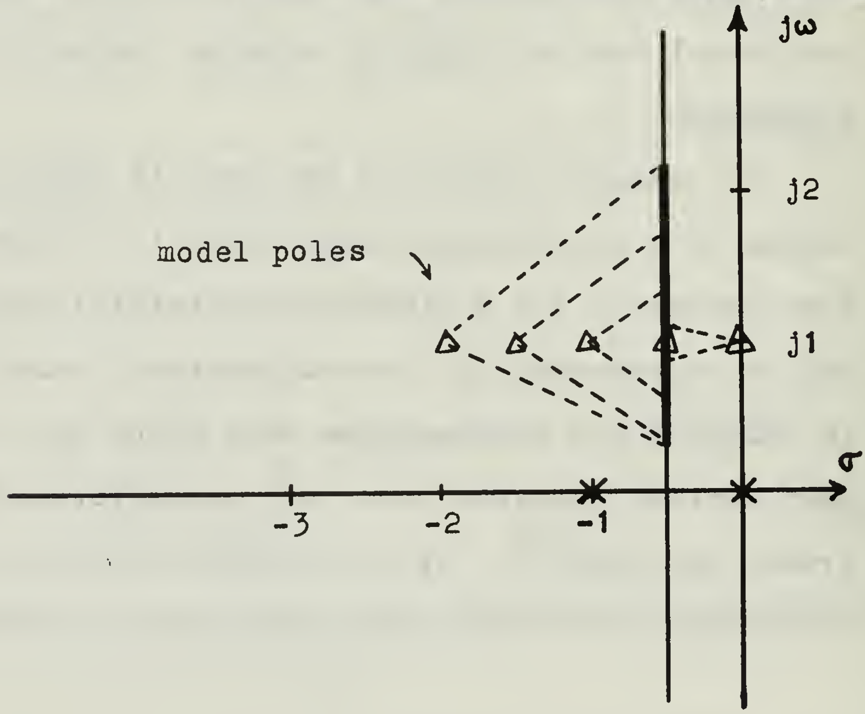

Fig. 11. Size of optimum regions for several models

$$J = x_1^2(0)\left[\frac{-22.4\,a_1a_2 + 16a_2^2 + a_1^3 + 16a_1 - 0.16\,a_1^2}{2a_1a_2}\right]$$

$$+\ 2x_1(0)x_2(0)\,\frac{(a_1-4)^2}{2a_1}\ +$$

$$x_2^2(0)\left[\frac{a_1a_2^2 - 5.6\,a_1a_2 - 0.16\,a_1 + a_1^2 + 16}{2a_1a_2}\right] \tag{33}$$

The shape of this cost surface will depend on the initial state used, as shown in equation (33) but the minimum value of this cost will always be zero and lie on the point (4,2.8), which corresponds to the model of this example. The cost surface is shown in Figures 12 and 13 for two different values of the initial state vector orientation. It can be seen that while the absolute minimum of the surface is fixed, the local minimum of the interception of the plane $a_2 = 1$ with these surfaces changes with initial conditions.

THIRD ORDER SYSTEM

Consider a third order linear control process characterized by the differential equation

$$\dddot{x} + 2\ddot{x} + a_2\dot{x} + a_1x = 0 \tag{34}$$

where the coefficient a_3 is constrained by the plant parameters to a value of 2. The specifications suggest a model

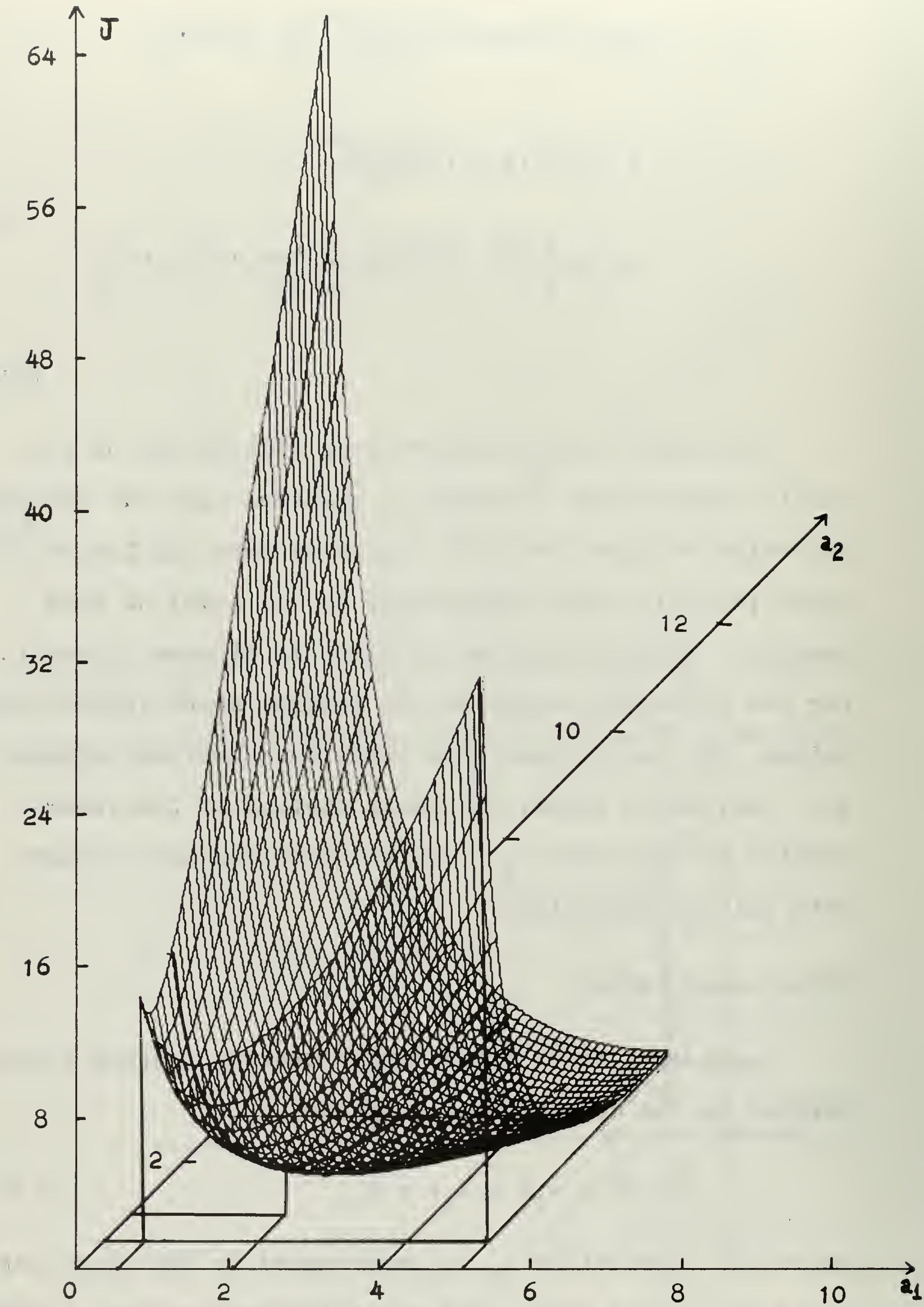

Fig. 12. Isometric projection of cost surface for a model characteristic equation $s^2 + 2.8s + 4 = 0$ and an initial state vector $\underline{x}^T(0) = (1 \quad 0)$

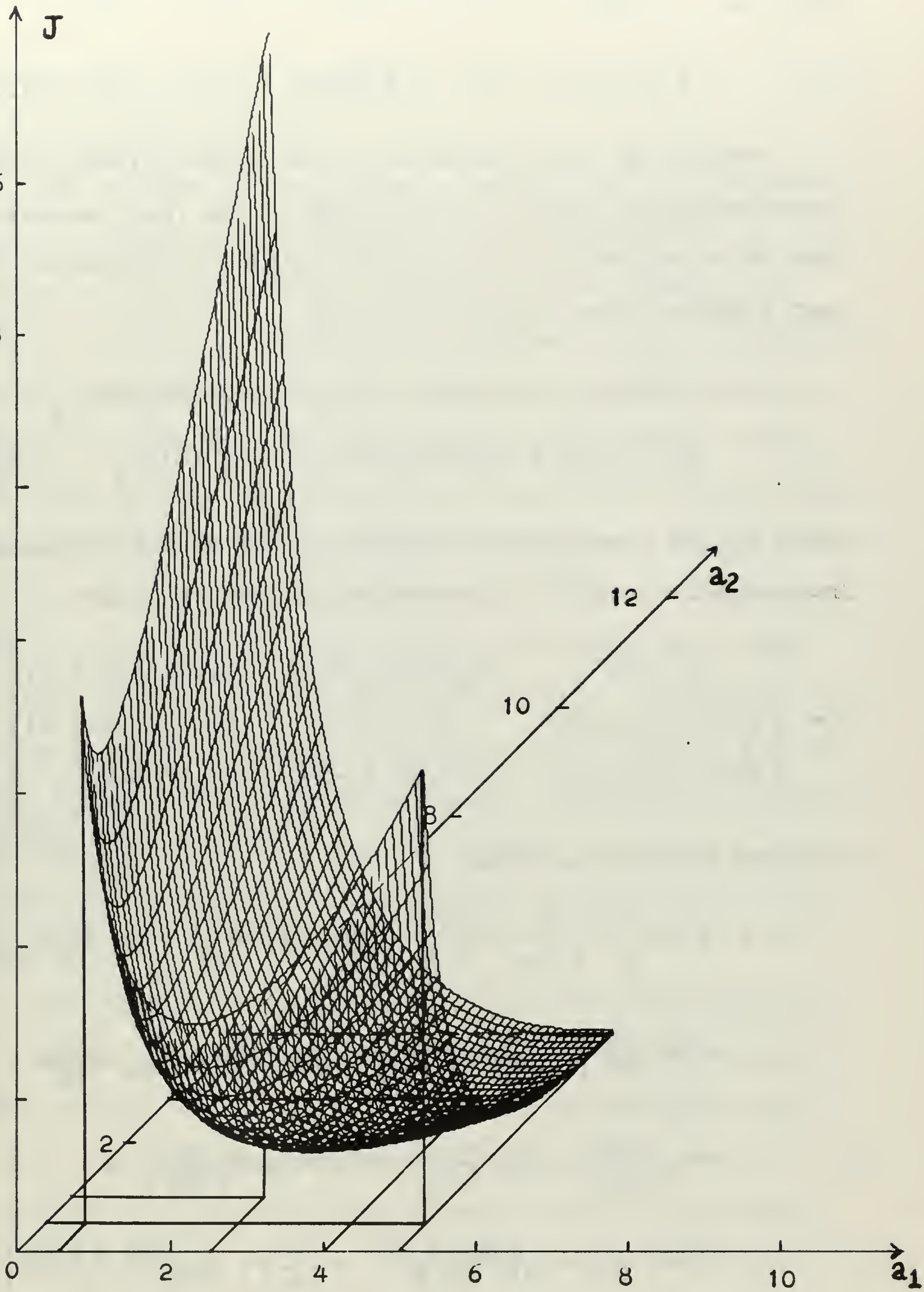

Fig. 13. Isometric projection of cost surface for a model characteristic equation $s^2 + 2.8s + 4 = 0$ and an initial state vector $\underline{x}^T(0) = (0.866 \quad 0.5)$

given by

$$\dddot{y} + 7.8\ddot{y} + 18\dot{y} + 20y = 0 \tag{35}$$

Because of the constraint in the actual system, these characteristics cannot be duplicated. The free parameters have to be adjusted to obtain the optimum performance. The cost function from equation (18) is

$$\begin{aligned} J = {} & x_1^2(0)p_{11} + 2x_1(0)x_2(0)p_{12} + 2x_1(0)x_3(0)p_{13} + \\ & x_2^2(0)p_{22} + 2x_2(0)x_3(0)p_{23} + x_3^2(0)p_{33} \end{aligned} \tag{36}$$

where p_{ij} is a particular element of the matrix P obtained from equation (19)[1]. The weighting matrix is now

$$B = \begin{bmatrix} (a_1-20)^2 & (a_1-20)(a_2-18) & (a_1-20)(2-7.8) \\ (a_2-18)(a_1-20) & (a_2-18)^2 & (a_2-18)(2-7.8) \\ (2-7.8)(a_1-20) & (2-7.8)(a_2-18) & (2-7.8)^2 \end{bmatrix} \tag{37}$$

The cost function is then

$$\begin{aligned} J = {} & x_1^2(0)\left[b_{11}\left(\frac{a_2}{2a_1} + \frac{a_3^2}{D}\right) - b_{12} + b_{22}\frac{a_1 a_3}{2D} - b_{13}\frac{a_1 a_3}{D} \right. \\ & \left. + b_{33}\frac{a_1^2}{2D}\right] + 2x_1(0)x_2(0)\left[-b_{13}\frac{a_2 a_3}{D} + b_{33}\frac{a_1 a_2}{2D} \right. \\ & \left. + b_{11}\frac{a_2 a_3^2}{2a_1 D} + b_{22}\frac{a_1}{2D}\right] + 2x_1(0)x_3(0)\left[\frac{b_{11}}{2a_1}\right] + \\ & x_2^2(0)\left[-b_{23} + b_{11}\left(\frac{a_2 a_3 + a_3^3}{2a_1 D} - \frac{1}{2a_1}\right) + b_{33}\left(\frac{a_2^2 + a_1 a_3}{2D}\right) \right. \\ & \left. - b_{13}\left(\frac{a_2 + a_3^2}{D}\right) + b_{22}\left(\frac{a_2 + a_3^2}{2D}\right)\right] + 2x_2(0)x_3(0)\left[b_{11}\frac{a_3^2}{2a_1 D} \right. \end{aligned}$$

$$+ b_{22} \frac{a_3}{2D} - b_{13} \frac{a_3}{D} + b_{33} \frac{a_1}{2D} \Big] + x_3^2(0) \Big[b_{11} \frac{a_3}{2a_1 D}$$

$$+ b_{22} \frac{1}{2D} - b_{13} \frac{1}{D} + b_{33} \frac{a_2}{2D} \Big] \tag{33}$$

where $D = a_2 a_3 - a_1$.

The effects of the initial state on the optimum pole location should be investigated for all points on the unit sphere in the state space. Because of the symmetry of equation (36) with respect to a diameter, only a hemisphere need be considered. For two degrees of freedom, the cost function corresponds to a three-dimensional cost surface. The shape of this subsurface will change with initial state and its corresponding minimum point will move around on the a_1-a_2 plane. The subsurface corresponding to a coefficient $a_3 = 7.8$ will have a minimum value of zero at the model values, independently of the initial state. A typical subsurface is shown in Figure 14 for the case $\underline{x}^T(0) = (-0.754\ 0.133\ 0.643)$. By computer searching techniques the optimum coefficients for this system were found for several initial conditions on the unit hemisphere to the right of the x-z plane.

The roots obtained from these optimum characteristic equations conglomerate in limited regions on the s-plane. These regions are shown in Figure 15. Observe that the problem of stability has been automatically solved. If the constraint on the actual system were released these optimum regions would collapse into points where the model poles are and the cost function would be identically zero for all

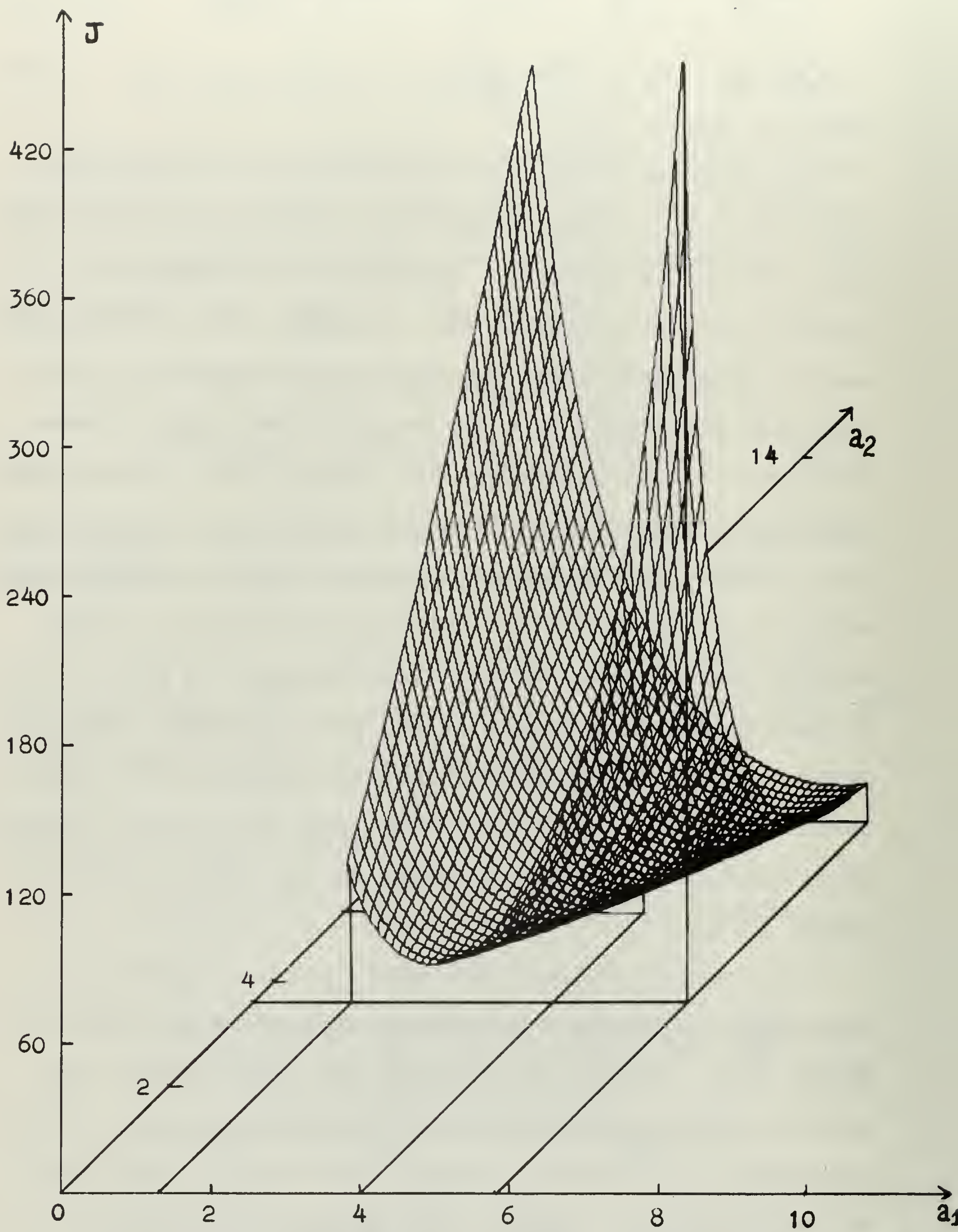

Fig. 14. Isometric projection of cost subsurface for a model characteristic equation $s^3+7.8s^2+18s+20=0$ and an actual system with a constraint $a_3=2$. The initial state vector is $\underline{x}^T(0)=(-0.754\ 0.133\ 0.643)$

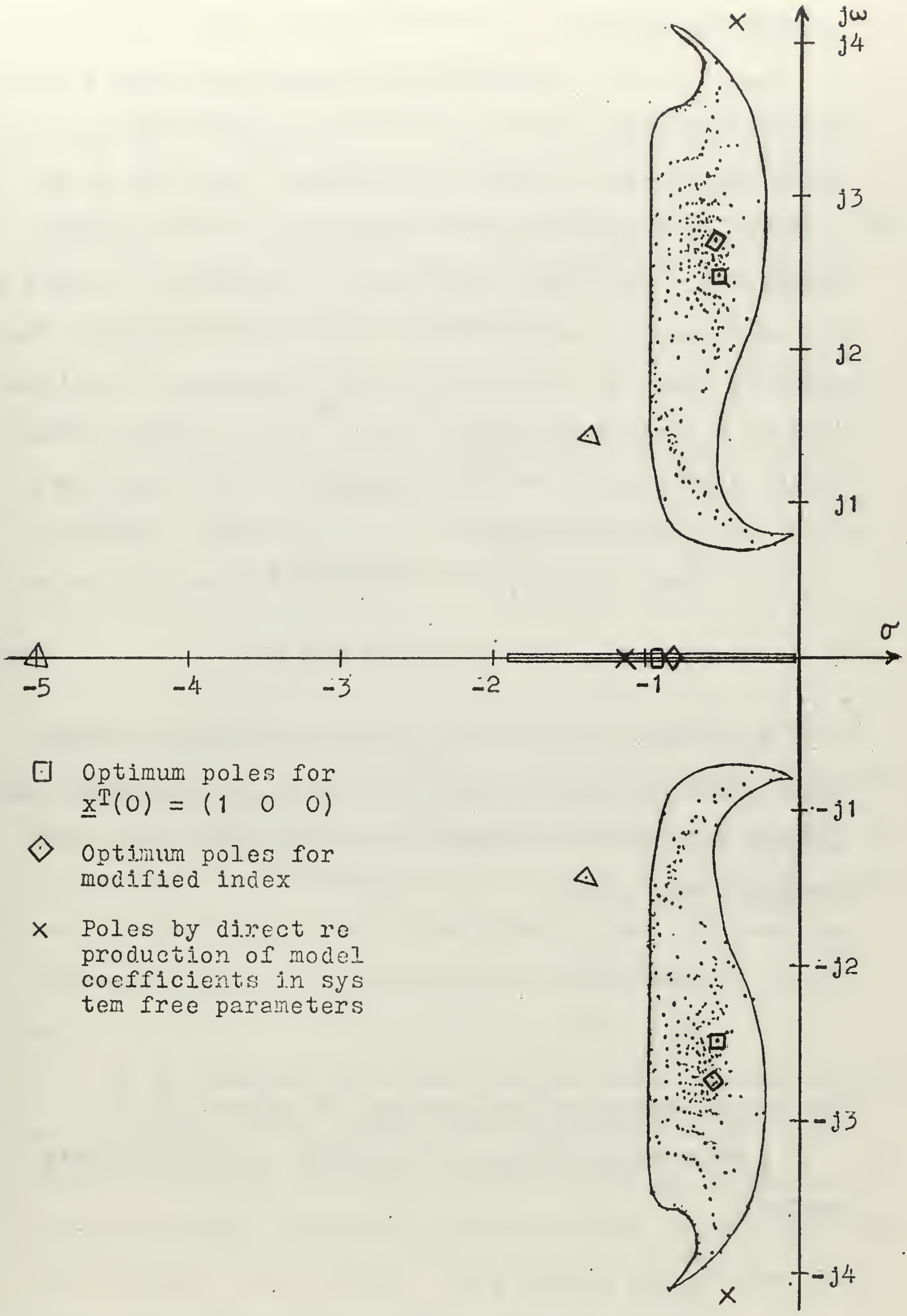

Fig. 15. Optimum pole regions for third order system with constraint in $a_3 = 2$

initial conditions.

Each set of optimum poles is associated with a value of the cost function, which is a minimum for the initial state from which this set was obtained. For any other initial state there will be a different set of optimum parameters. The final selection of parameters is based on the statistical characteristics of the disturbance. The modified index is obtained from an extension of equation (31) to a three-dimensional space. For a uniform probability distribution over the hemisphere this index is

$$J_{mod} = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} J(\theta, a_2, a_1)\, d\theta\, d\phi$$

$$= \frac{1}{4}\left(p_{11} + p_{22} + 2p_{33}\right) + \frac{2}{\pi} p_{13} \qquad (39)$$

This modified index is independent of the initial state. The minimum average cost function, and hence the minimax solution is obtained for this example with the optimum coefficients

$$a_1 = 6.5$$

$$a_2 = 8.7$$

The corresponding poles are shown in Figure 15.

A reasonable choice is to assume an initial state vector

$$\underline{x}^T(0) = (1 \;\; 0 \;\; 0)$$

The optimum set of poles for this case is shown in Figure

15. The cost subsurface is shown in Figure 16. This set of initial conditions resembles the widely used step function in control systems design.

A worthwhile observation is that the available freedom in the system parameters could be used to reproduce exactly some of the corresponding model coefficients. Consider the actual system and model differential equations as given in equations (34) and (35)

$$\dddot{x} + 2\ddot{x} + a_2\dot{x} + a_1 x = 0 \tag{34}$$

$$\dddot{y} + 7.8\ddot{y} + 18\dot{y} + 20y = 0 \tag{35}$$

The direct reproduction of the model coefficients in the actual system will yield

$$\dddot{x} + 2\ddot{x} + 18\dot{x} + 20x = 0 \tag{40}$$

The corresponding poles are close to the optimum regions obtained by the previous method. They are shown in Figure 15. This possibility in parameter selection is limited by the stability requirements $a_1 > 0$, $a_2 > 0$, $a_2 a_3 - a_1 > 0$. Whenever possible, this criterion gives a good, time-saving procedure if the third requirement for stability is well met.

It is interesting to see how the optimum poles migrate as the initial state vector sweeps the normalized state space. A mapping of a unit circle on the x_1-x_2 plane in the state space onto the s-plane is shown in Figure 17 for this example. The optimum poles return to the same point after the initial conditions vector turns 180°.

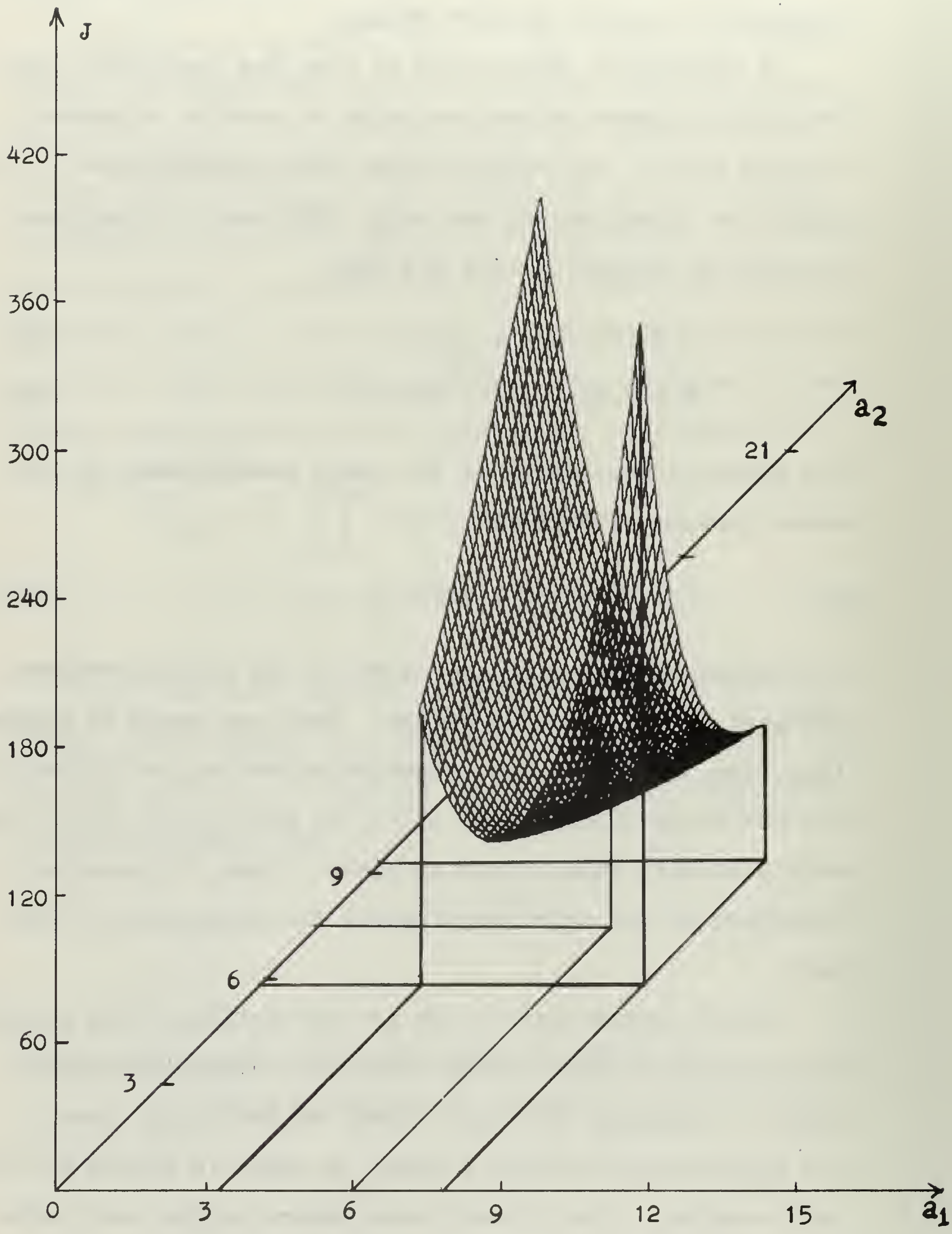

Fig. 16. Isometric projection of cost subsurface. Model and actual system as in Figure 14. Initial state vector $\underline{x}^T(0) = (1 \;\; 0 \;\; 0)$

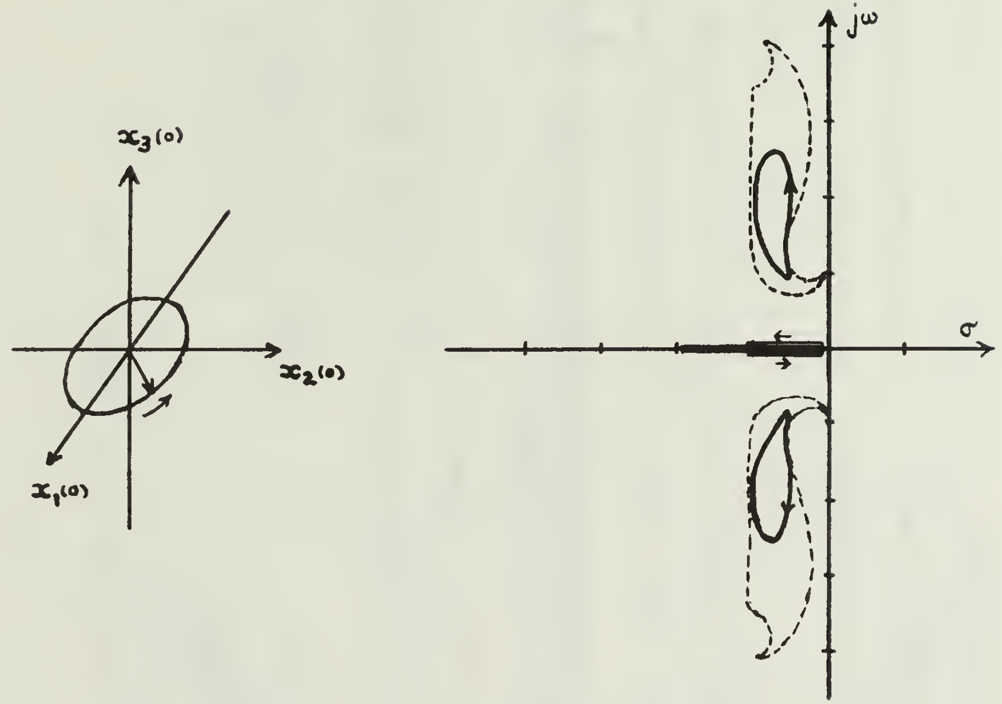

Fig. 17. Mapping of the unit circle on the x_1-x_2 plane onto the s-plane

Consider now that for the same performance specifications, the constraint on the actual system is somewhat less stringent. Assume that the system's characteristic equation is

$$s^3 + 6s^2 + a_2 s + a_1 = 0 \tag{41}$$

The model poles will now be closer to the attainable regions of the given system and consequently, the results of the optimization procedure will be included in smaller regions. These regions are shown in Figure 18. There is also a net displacement of these regions to the left and now the complex conjugate model poles are included. The set of poles obtained by the mere duplication of model coefficients in the free terms is also shown.

Using the modified index of equation (39) for a uniform probability on the initial state, the final selection is

$$a_1 = 16.3$$

$$a_2 = 15.2$$

The optimum poles are shown in Figure 18.

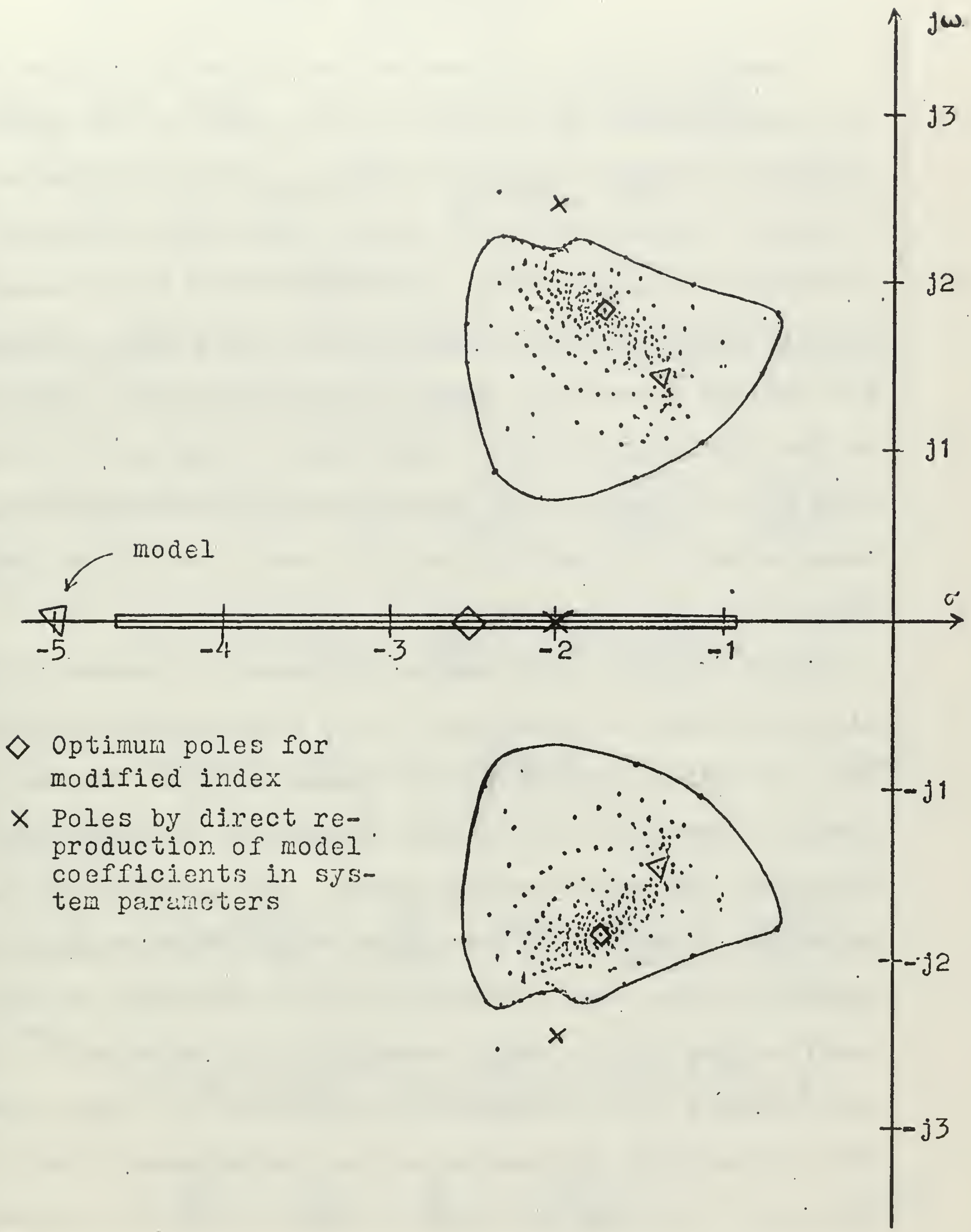

Fig. 18. Optimum pole regions for third order system with constraint in $a_3 = 6$

CHAPTER IV

EVALUATION OF RESULTS

Because of the constraints on the actual system and the requirements for stability, the poles of the system transfer function are restricted to certain regions on the s-plane. The optimum pole regions have a size, shape, and position according to the proximity of the model poles to the allowable regions. The selection of system parameters for optimum design was based on the statistical properties of the disturbing initial conditions. That set of poles from the optimum regions which gives the minimum average index under all possible initial conditions is the desired solution to the problem.

The index of performance developed in Chapter II was given the form of equation (1) for mathematical convenience. The two indices differ in the description of the weighting matrix. However, the optimum parameters obtained are identical for every initial state. The elements of the Q matrix in equation (1) were chosen to place the absolute minimum of the cost function at the coordinates of the coefficients of the model characteristic equation[1] while the elements of the B matrix in equation (17) came directly from the general expression of the performance index of equation (12) when expressed in matrix form for mathematical manipulation.

A comparison between the two indices is done for the second order system. The weighting matrices, Q[1] and B,

are

$$Q = \begin{bmatrix} 1+\lambda a_1^2 & \lambda a_1 a_2 \\ \lambda a_1 a_2 & \alpha_2 + \lambda a_2^2 \end{bmatrix}$$

where $\lambda = 1/g_1^2$ and $\alpha_2 = (g_2^2 - 2g_1)/g_1^2$

$$B \quad \begin{bmatrix} (g_1 - a_1)^2 & (g_1 - a_1)(g_2 - a_2) \\ (g_2 - a_2)(g_1 - a_1) & (g_2 - a_2)^2 \end{bmatrix}$$

The expanded form of the cost function given in equation (2) is, by use of equation (22)

$$J_Q = x_1^2(0)\left[\frac{a_2}{2a_1} + \frac{1}{2a_2} + \frac{a_1^2}{2g_1^2 a_2} + \frac{a_1 g_2^2}{2g_1^2 a_2} - \frac{a_1}{g_1 a_2}\right]$$

$$+ 2x_1(0)x_2(0)\left[\frac{1}{2a_1} + \frac{a_1}{2g_1^2}\right] + x_2^2(0)\left[\frac{g_2^2}{2g_1^2 a_2} - \frac{1}{g_1 a_2}\right.$$

$$\left. + \frac{a_2}{2g_1^2} + \frac{1}{2a_1 a_2} + \frac{a_1}{2g_1^2 a_2}\right] \qquad (44)$$

Similarly

$$J_B = x_1^2(0)\left[\frac{g_1^2 a_2}{2a_1} + \frac{a_1^2}{2a_2} - \frac{g_1 a_1}{a_2} + \frac{g_1^2}{2a_2} + \frac{a_1 g_2^2}{2a_2}\right]$$

$$+ 2x_1(0)x_2(0)\left[\frac{a_1}{2} - g_1 + \frac{g_1^2}{2a_1}\right] + x_2^2(0)\left[\frac{a_2}{2} - g_2\right.$$

$$\left. + \frac{g_2^2}{2a_2} + \frac{a_1}{2a_2} - \frac{g_1}{a_2} + \frac{g_1^2}{2a_1 a_2}\right] \qquad (45)$$

where J_Q = index of performance with matrix Q

J_B = index of performance with matrix B

Equations (44) and (45) are related by

$$g_1^2 J_Q - J_B = x_1^2(0)g_1 g_2 + 2x_1(0)x_2(0)g_1 + x_2^2(0)g_2 \qquad (46)$$

For any given initial conditions, the right hand side of equation (46) is a constant related to the model coefficients and the particular set of initial conditions. Thus, in general, both cost functions differ by a constant factor, which is the square of the model coefficient g_1, and an additive constant given by equation (46). As a consequence, the optimum coefficients of the actual system obtained by the minimization of these two indices are identical. This is true for systems of any order. However, the algorithm to determine the B matrix is simpler than that for the Q matrix.

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13. ABSTRACT

Fixed configuration feedback control often forms the basis for the design of a linear control system. The design freedom is limited to the adjustment of the free parameters of the system. Analytical methods prove to be valuable in these circumstances and the aim is the optimization of a selected index of performance. A suggested index is developed for a regulator system from a comparison of the dynamics of the subject system with those of a desired model along a transient state space trajectory. The result is an optimal system dependent on the initial state chosen for the transient response. The effects of the initial state on the optimal system are investigated and the final selection is based on statistical considerations.

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